Little (but Hard) Theorems About Big Systems: Some Case Studies

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Lecture 3

Definitions

When using a formal system it is necessary to be able to *extend* the theory with the introduction of new concepts.

For example, if we cannot introduce the new functions `rev` or `rev1`, how can we prove anything about them?

But it is risky to add new axioms to a formal system in an undisciplined way.

How do we know the system is consistent after adding a new axiom?

A *Definitional Principle* permits one to extend a formal system without risk.

Theorem. `(p x)` $\rightarrow$ `(q x)`.

Proof:

Definition: `(defun r (x) ...)`

Lemma 1: `(p x)` $\rightarrow$ `(r x)`.

Lemma 2: `(r x)` $\rightarrow$ `(q x)`.

Q.E.D.

But how do you know `(p x)` $\rightarrow$ `(q x)` is a theorem in the original theory? You added an axiom!

Definitions

$\texttt{(defun}\ f\ \ (v_1\ \ \ldots\ \ v_n)\ \ \beta\texttt{)}$

introduces the new axiom

$\forall v_1 \ldots v_n\ \texttt{:}\ \ (f\ \ v_1 \ldots v_n)\ \texttt{=}\ \beta$

provided the definition is *admissible*.

Such definitions are *conservative*: the only new theorems are ones involving the new symbol.

(`defun` f (v_1 ... v_n) β)

is admissible iff

1. f is a new function symbol
2. the v_i are distinct variable symbols
3. β is a term that contains no (free) variable symbols other than the v_i

```
(defun car (x) x)   ; Violates 1

(defun f (x x) x)   ; Violates 2

(defun g (x) y)     ; Violates 3
```

Theorem: $2=7$

Proof:

We have the axiom $\forall$`x: (g x)=y`.

$(\mathtt{g}\ \mathtt{0}) = \mathtt{2}$

$(\mathtt{g}\ \mathtt{0}) = \mathtt{7}$

$\mathtt{2} = \mathtt{7}$

Q.E.D.

What is the harm in non-terminating functions?

```
(defun f (x)
  (if (equal x 1)
      nil
      (cons nil (f (- x 1)))))
```

Lemma:

$\texttt{(natp n)} \wedge \texttt{x} < \texttt{1} \rightarrow \texttt{n} < \texttt{(len (f x))}.$

Theorem: `(len (f 0)) < (len (f 0))`.

Admissiblity (continued)

4. There is a natural number measure ($\mathtt{m}\ v_1 \ldots v_n$) such that for every recursive call ($\mathtt{f}\ \delta_1 \ldots \delta_n$) in β and its governing tests τ:

Measure Theorem:

$$
\begin{array}{l}
(\mathtt{implies}\ \tau \\
\qquad\qquad (\mathtt{<}\ (\mathtt{m}\ \delta_1\ \ldots\ \delta_n) \\
\qquad\qquad\qquad (\mathtt{m}\ v_1\ \ldots\ v_n)))
\end{array}
$$

```
(defun app (x y)
  (if (consp x)
      (cons (car x)
            (app (cdr x) y))
     y))
```

`(consp x)` governs `(app (cdr x) y)`.

Measure Theorem:

```
(implies (consp x)
         (< (m (cdr x) y)
            (m x y)))
```

Measure Theorem:

```
(implies (consp x)
         (< (m (cdr x) y) (m x y)))
```

where we measure the length of `x`

```
(defun m (x y) (len x))
```

or the "tree size"

```
(defun m (x y) (acl2-count x))
```

Elaboration: `m` may return an ordinal, in which case, < should be replaced by `o<`.

Note: ACL2 represents the ordinals up to ϵ_0.

$$\epsilon_0 = \omega^{\omega^{\omega^{\cdots}}}$$

Examples (ordered by o<)

0	0
1	1
2	2
ω	((1 . 1) . 0)
$\omega \times 2$	((1 . 2) . 0)
$\omega \times 2 + 23$	((1 . 2) . 23)
$\omega^2 \times 3 + \omega \times 7 + 19$	((2 . 3) (1 . 7) . 19)
ω^ω	((((1 . 1) . 0) . 1) . 0)

ACL2 provides support for ordinal arithmetic.

$$\omega^2 \times 3 + \omega \times 7 + 19$$

```
(o+ (o* (o^ (omega) 2) 3)
    (o* (omega) 7)
    19)
=
((2 . 3) (1 . 7) . 19)
```

Induction

To prove `(`ψ `x y)` by induction on `x` prove:

Base Case:

```
(implies (not (consp x)) (ψ x y))
```

Induction Step:

```
(implies (and (consp x)
              (ψ (car x) α1)
              (ψ (cdr x) α2))
         (ψ x y))
```

where the α_i are arbitrary terms.

Elaborations

You may, of course, omit an induction hypothesis.

Induction Step:

$$
\begin{array}{l}
\texttt{(implies (and (consp x)} \\
\qquad\qquad\qquad\quad \texttt{(}\psi\ \texttt{(cdr x)}\ \alpha_2\texttt{))} \\
\qquad\qquad\quad \texttt{(}\psi\ \texttt{x y))}
\end{array}
$$

You may have multiple α_i.

Induction Step:

```
(implies (and (consp x)
              (ψ (car x) α1)
              (ψ (cdr x) α2)
              (ψ (cdr x) α3))
         (ψ x y))
```

$$(\text{implies (and (consp x)}\; (\psi\ \texttt{(car x)}\ \alpha_1)\; (\psi\ \texttt{(cdr x)}\ \alpha_2)\; (\psi\ \texttt{(cdr x)}\ \alpha_3))\; (\psi\ \texttt{x y}))$$

Instead of `(car x)` and `(cdr x)` you may use any terms whose values are smaller than `x`'s when the "test" `(consp x)` holds.

Induction Step:

$$
\begin{array}{l}
\texttt{(implies (and (consp x)} \\
\qquad\qquad\qquad\qquad\texttt{(}\psi\ \texttt{(lo x)}\ \alpha_1\texttt{)} \\
\qquad\qquad\qquad\qquad\texttt{(}\psi\ \texttt{(hi x)}\ \alpha_2\texttt{))} \\
\qquad\qquad\quad\texttt{(}\psi\ \texttt{x y))}
\end{array}
$$

`(lo` x`)` $=$ list $e \in x : e <$ `(car x)`.

`(hi` x`)` $=$ list $e \in x : e >$ `(car x)`.

Induction Principle To prove ψ, let τ be a term and let $\sigma_1, \sigma_2, \ldots$ be variable-to-term substitutions.

Base Case:

$$\texttt{(implies (not } \tau\texttt{)}\ \psi\texttt{)}$$

Induction Step:

$$\texttt{(implies (and } \tau\ \psi/\sigma_1\ \ldots\ \psi/\sigma_n\texttt{)}\ \psi\texttt{)}$$

provided

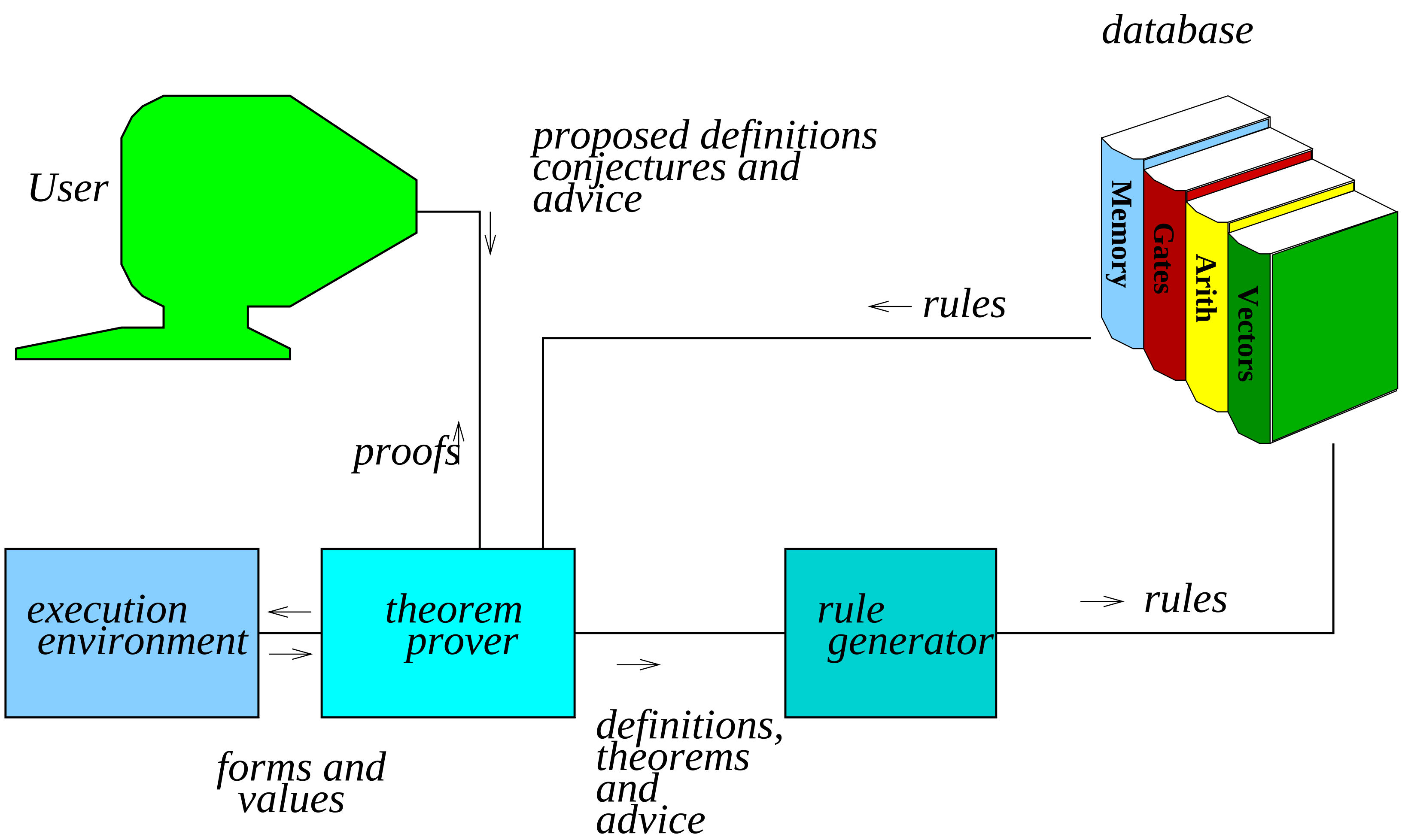
there is a term m such that

`(o-p` m`)`, and

for each $1 \leq i \leq n$

```
(implies τ
         (o< m/σ_i m))
```

are theorems.


database
User
proposed definitions
conjectures and
advice
Memory
Gates
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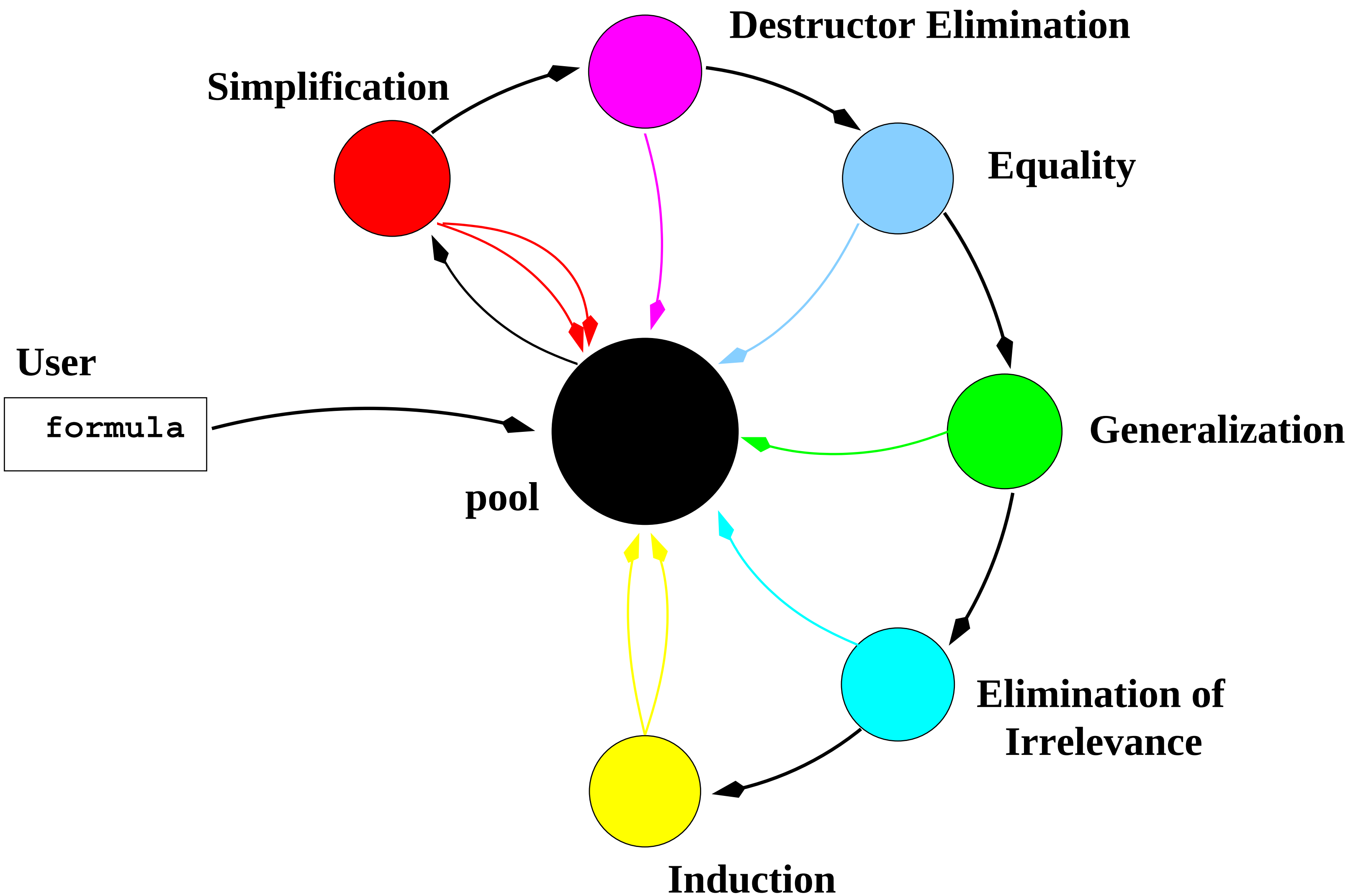
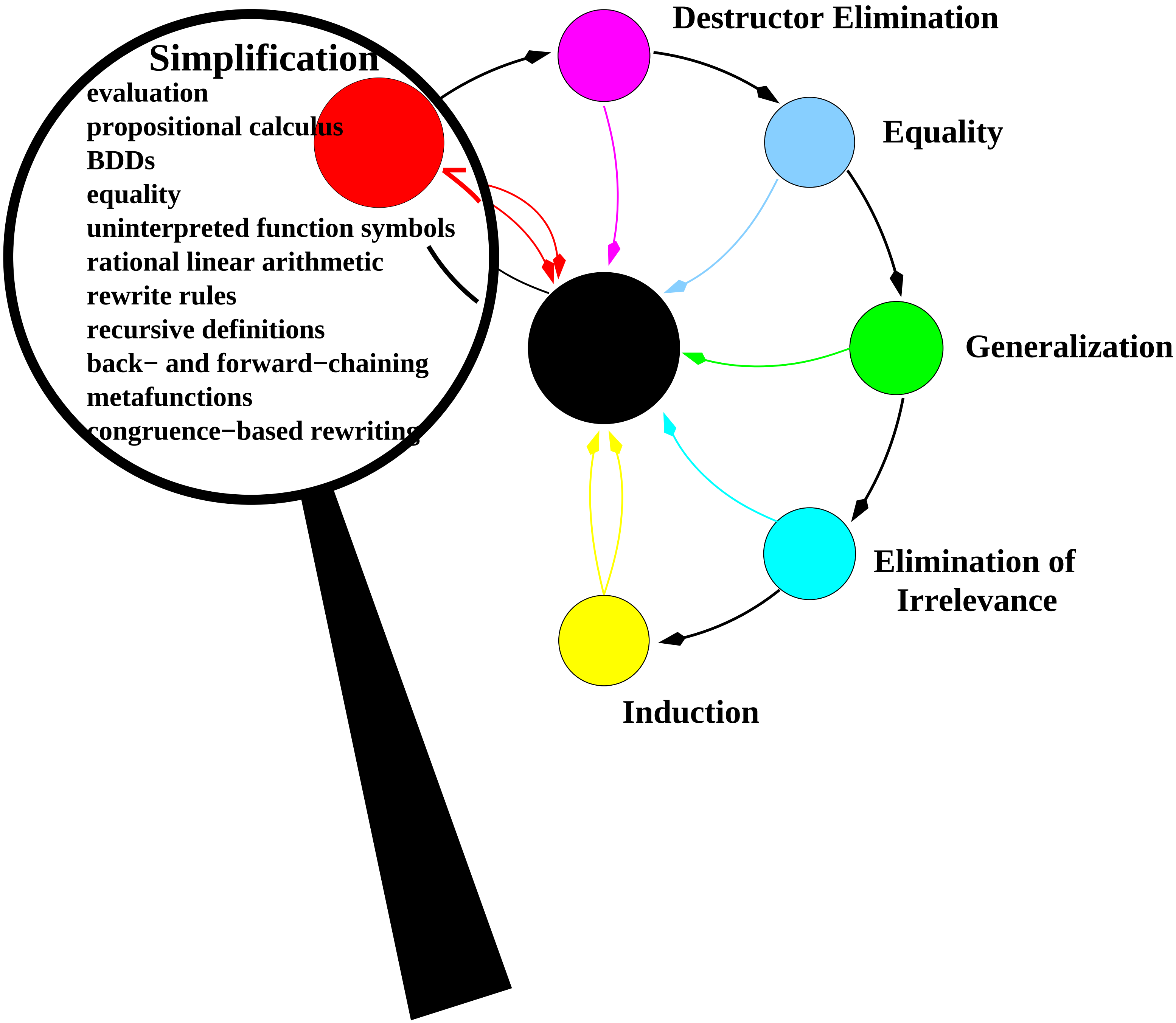

Destructor Elimination
Simplification
Equality
User
formula
Generalization
pool
Elimination of Irrelevance
Induction


Simplification
evaluation
propositional calculus
BDDs
equality
uninterpreted function symbols
rational linear arithmetic
rewrite rules
recursive definitions
back− and forward−chaining
metafunctions
congruence−based rewriting
Destructor Elimination
Equality
Generalization
Elimination of Irrelevance
Induction

Demo

Note

I will make available my solutions to the `twins` problem, the `subp` problem, and the `Hanoi` problem.

Next time I will begin to present larger systems.