Constructive Systems: Models and Applications

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The central difference between working in constructive rather than classical mathematics is the meaning of existence. It came explicitly to the fore when Zermelo could not exhibit the well-ordering of the reals he claimed to have proved existed. Brouwer, the originator of intuitionism, rejected proofs by contradiction of the existence of objects because they do not supply the object they purportedly established. He held that no sensible meaning could be attached to the phrase "there exists" other than "we can find".

The history of the development of mathematics can in part be seen as a search for more general and flexible data structures. First one had the integers, then the rational, real and complex numbers, the general concept of function, and eventually arbitrary sets. Set theory is identified with rigor and has earned the status of providing a full scale system for formalizing mathematics. On the other hand, set theory has a reputation for being non-computational and nonconstructive. This is certainly true for classical set theory but there is nothing intrinsically nonconstructive about sets. In computer science, constructive formal systems based on type theory or on the Curry-Howard isomorphism have shown their utility for program development and extraction of algorithms from proofs.

This course will be concerned with type-free set theories based on intuitionistic logic which turn out to be closer than one might expect to the typed systems such as Martin-Löf's type theory, Coquand and Huet's Theory of Constructions, etc. Intuitionistic set theory constitutes a major site of interaction between constructivism, set theory, proof theory, type theory, topos theory and computer science. Particular emphasis will be given to interpretations and semantics that allow for extraction of algorithms from proofs, i.e. notions of realizability. Further interpretations to be surveyed include Kripke, forcing, Heyting-valued, and categorical models.

There is a widespread impression that the prize for relinquishing classical logic is high. However, the loss of certain comforting theorems of classical mathematics can bring forth considerable profits. Intuitionistic logic allows for axiomatic freedom in that one can adopt new axioms that are true in realizability models but outrageously false classically. Examples of such principles are provided by the so-called Church's thesis which asserts that all functions from the integers to the integers are computable and by Brouwer's Theorem which asserts that all functions from the reals to the reals are continuous. Time permitting, a strand of the course will be devoted to applications of realizability models to Scott's information systems and synthetic domain theory.

References

1. P. Aczel, M. Rathjen. *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001
   `http://www.ml.kva.se/preprints/archive2000-2001.php`
   (Most comprehensive text on constructive Zermelo-Fraenkel set theory.)

2. M. Beeson. *Foundations of Constructive Mathematics*, Springer-Verlag, 1985
   (Encyclopaedic text on constructive theories.)

3. K. Kunen. *Set Theory: An introduction to independence proofs*, North-Holland, Amsterdam, 1980
   (Excellent book on classical set theory.)

4. A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics, Volumes I,II*, North Holland, Amsterdam, 1988
   (The book to learn about intuitionism and realizability interpretations.)