1.6 Exercises

Exercise 1 *Use Tseitin's transformation to convert* $x + (y \cdot (\overline{z} \oplus x))$ *into CNF.*

Solution By introducing the following fresh variables

$$\overbrace{x + \underbrace{(y \cdot \underbrace{((\overbrace{\overline{z} \cdot \overline{x}}^{p}) + (\overbrace{z \cdot x}^{q}))}_{u})}_{v}}^{w}$$

we obtain the formula

$$w \cdot (q \leftrightarrow (z \cdot x)) \cdot (p \leftrightarrow (\overline{z} \cdot \overline{x})) \cdot (u \leftrightarrow (p + q)) \cdot (v \leftrightarrow (y \cdot u)) \cdot (w \leftrightarrow (x + v))$$

We can now apply the rules

$$a \leftrightarrow (b + c) \equiv (\overline{b} + a) \cdot (\overline{c} + a) \cdot (\overline{a} + b + c) \tag{1.6}$$
$$a \leftrightarrow (b \cdot c) \equiv (\overline{a} + b) \cdot (\overline{a} + c) \cdot (\overline{b} + \overline{c} + a) \tag{1.7}$$

and get

$$\begin{aligned} w \cdot (\overline{q} + z) \cdot (\overline{q} + x) \cdot (\overline{z} + \overline{x} + q) \cdot (\overline{p} + \overline{z}) \cdot (\overline{p} + \overline{x}) \cdot (z + x + p) \cdot \\ (\overline{p} + u) \cdot (\overline{q} + u) \cdot (\overline{u} + p + q) \cdot (\overline{y} + v) \cdot (\overline{u} + v) \cdot (\overline{v} + y + u) \cdot \\ (\overline{x} + w) \cdot (\overline{v} + w) \cdot (\overline{w} + x + v) \end{aligned}$$

Exercise 2 *Follow the scheme in Table 1.2 in Section 1.2.1 to derive the Tseitin clauses that characterise the* n*-ary Boolean formulas* $(y_1 + y_2 + \cdots + y_n)$ *and* $(y_1 \cdot y_2 \cdot \cdots \cdot y_n)$.

Solution

- Disjunction:

$$\begin{aligned} & x \leftrightarrow (y_1 + y_2 + \cdots + y_n) \\ \equiv\ & (x \rightarrow (y_1 + y_2 + \cdots + y_n)) \cdot ((y_1 + y_2 + \cdots + y_n) \rightarrow x) \\ \equiv\ & (\overline{x} + y_1 + y_2 + \cdots + y_n) \cdot ((y_1 \rightarrow x) \cdot (y_2 \rightarrow x) \cdots (y_n \rightarrow x)) \\ \equiv\ & (\overline{x} + y_1 + y_2 + \cdots + y_n) \cdot (\overline{y}_1 + x) \cdot (\overline{y}_2 + x) \cdots (\overline{y}_n + x) \end{aligned}$$

- Conjunction:

$$\begin{aligned} & x \leftrightarrow (y_1 \cdot y_2 \cdot \cdots \cdot + y_n) \\ \equiv\ & (x \rightarrow (y_1 \cdot y_2 \cdot \cdots \cdot y_n)) \cdot ((y_1 \cdot y_2 \cdot \cdots \cdot y_n) \rightarrow x) \\ \equiv\ & ((\overline{x} + y_1) \cdot (\overline{x} + y_2) \cdot \cdots \cdot (\overline{x} + y_n)) \cdot \left(\overline{(y_1 \cdot y_2 \cdot \cdots \cdot y_n)} + x\right) \\ \equiv\ & ((\overline{x} + y_1) \cdot (\overline{x} + y_2) \cdot \cdots \cdot (\overline{x} + y_n)) \cdot (\overline{y}_1 + \overline{y}_2 + \cdots + \overline{y}_n + x) \end{aligned}$$

Exercise 3 *Which of the Boolean formulae below are satisfiable, and which ones are unsatisfiable?*

1. $x + x \cdot y$
2. $\overline{(x \cdot (x \rightarrow y)) \rightarrow y}$
3. $\overline{x \cdot ((x \rightarrow y) \rightarrow y)}$

Convert the formulae that are unsatisfiable into conjunctive normal form (either using Tseitin's transformation or the propositional calculus) and construct a resolution refutation proof.

Solution

- satisfiable: 1, 3
- unsatisfiable: 2

$$\begin{aligned}\overline{(x \cdot (x \rightarrow y)) \rightarrow y} &\equiv \overline{\overline{(x \cdot (\overline{x} + y))} + y}\\ &\equiv (x) \cdot (\overline{x} + y) \cdot (\overline{y})\end{aligned}$$

Resolution proof:

$$\mathrm{Res}((\overline{y}), \mathrm{Res}((x), (\overline{x} + y), x), y) \equiv \Box$$

Exercise 4 *Construct a resolution refutation graph for the following unsatisfiable formula:*

$$y_1 \cdot y_2 \cdot y_3 \;\cdot\; (\overline{y}_1 + x) \;\cdot\; (\overline{y}_2 + \overline{x} + z) \;\cdot\; (\overline{y}_3 + \overline{z})$$

Solution The resolution graph for Exercise 4 is shown in Figure 1.17.

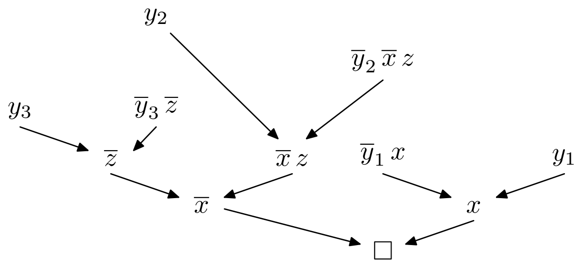

Figure 1.17: Resolution graph for Exercise 4

Exercise 5 *Apply the rules of the Davis-Putnam procedure (outlined in Section 1.3.3) to the following formula until you obtain an equi-satisfiable formula that cannot be reduced any further:*

$$y_1 \cdot y_2 \cdot (\overline{y}_1 + x + \overline{z}) \;\cdot\; (\overline{y}_2 + \overline{x} + z) \;\cdot\; (y_3 + \overline{z}) \;\cdot\; y_4$$

Solution We perform the following steps:

Step	**Rule**	**Formula**
1	*1-literal-rule* on y_1	$y_2 \cdot (x+\overline{z}) \cdot (\overline{y}_2+\overline{x}+z) \cdot (y_3+\overline{z}) \cdot y_4$
2	*1-literal-rule* on y_2	$(x+\overline{z}) \cdot (\overline{x}+z) \cdot (y_3+\overline{z}) \cdot y_4$
3	*Affirmative-negative*	$(x+\overline{z}) \cdot (\overline{x}+z)$
4	*Resolution* on x	$(z+\overline{z})$

The resulting formula $(z+\overline{z})$ is a tautology and cannot be eliminated by any of the Davis-Putnam rules. Accordingly, the original formula must be satisfiable.

Exercise 6 *Apply the Davis-Putnam-Logeman-Loveland (DPLL) procedure (described in Section 1.3.4) to the following formula:*

$$y_1 \cdot y_2 \cdot (\overline{y}_1+x+z) \cdot (\overline{y}_2+\overline{x}+z) \cdot (y_3+\overline{z}) \cdot (\overline{y}_3+\overline{z})$$

Solution Table 1.4 shows one possible scenario. Note that there is no value of x that satisfies the formula. The reader may verify that choosing a decision variable other than x in the third step also yields a contradiction.

Partial Assignment	**Clauses**
$\{y_1 \mapsto 1\}$	$(y_2)\,(x\,z)\,(\overline{y}_2\,\overline{x}\,z)\,(y_3\,\overline{z})\,(\overline{y}_3\,\overline{z})$
$\{y_1 \mapsto 1, y_2 \mapsto 1\}$	$(x\,z)\,(\overline{x}\,z)\,(y_3\,\overline{z})\,(\overline{y}_3\,\overline{z})$
No more implications, we guess $x \mapsto 1$	
$\{y_1 \mapsto 1, y_2 \mapsto 1, x \mapsto 1\}$	$(z)\,(y_3\,\overline{z})\,(\overline{y}_3\,\overline{z})$
$\{y_1 \mapsto 1, y_2 \mapsto 1, x \mapsto 1, z \mapsto 1\}$	$(y_3)\,(\overline{y}_3)$
$\{y_1 \mapsto 1, y_2 \mapsto 1, x \mapsto 1, z \mapsto 1, y_3 \mapsto 1\}$	0
Contradiction, we have to revert $x \mapsto 1$	
$\{y_1 \mapsto 1, y_2 \mapsto 1, x \mapsto 0\}$	$(z)\,(y_3\,\overline{z})\,(\overline{y}_3\,\overline{z})$
$\{y_1 \mapsto 1, y_2 \mapsto 1, x \mapsto 0, z \mapsto 1\}$	$(y_3)\,(\overline{y}_3)$
$\{y_1 \mapsto 1, y_2 \mapsto 1, x \mapsto 0, z \mapsto 1, y_3 \mapsto 1\}$	0
Contradiction, no more decisions to undo	

Table 1.4: Assignment trail for Exercise 6

Exercise 7 *Simulate the* conflict-driven clause learning *algorithm presented in Section 1.3.5 on the following formula:*

$$\overbrace{(\overline{x}+\overline{y}+\overline{z})}^{C_0} \cdot \overbrace{(\overline{x}+\overline{y}+z)}^{C_1} \cdot \overbrace{(\overline{x}+y+\overline{z})}^{C_2} \cdot \overbrace{(\overline{x}+y+z)}^{C_3} \cdot$$

$$\overbrace{(x+\overline{y}+\overline{z})}^{C_4} \cdot \overbrace{(x+\overline{y}+z)}^{C_5} \cdot \overbrace{(x+y+\overline{z})}^{C_6} \cdot \overbrace{(x+y+z)}^{C_7}$$

Solution It is obvious that one has to make at least two decisions before one of the clauses becomes unit. If we start with the decisions $x@1$ and $y@2$, we obtain the implication graph in Figure 1.18.

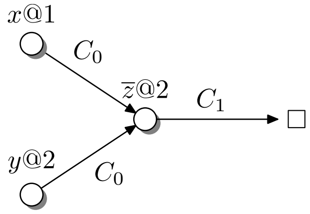

Figure 1.18: First implication graph arising in Exercise 7

By means of resolution (c.f. Section 1.3.6) we obtain the conflict clause $C_8 \equiv \text{Res}(C_0, C_1, z) \equiv (\overline{x} + \overline{y})$. We revert all decisions up to (but excluding) level 1, which is the second-highest decision level occurring in C_8. The clause C_8 is unit under the assignment $x@1$, thus implying the assignment $\overline{y}@1$. We obtain the implication graph in Figure 1.19. Again, there is a conflict.

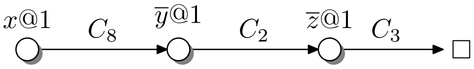

Figure 1.19: Second implication graph arising in Exercise 7

The resulting conflict clause is $C_9 \equiv \text{Res}(C_8, \text{Res}(C_2, C_3, z), y) \equiv (\overline{x})$, forcing us to revert to decision level zero and set x to $\mathsf{0}$. Under this assignment, none of the clauses is unit and we have to make a choice for either y or z. If we choose $y@1$, the clause C_4 becomes assertive and forces us to assign $\mathsf{0}$ to z. This assignment, however, is in conflict with C_5, and by means of resolution we obtain the conflict clause $C_{10} \equiv \text{Res}(C_4, C_5, z) \equiv (x + \overline{y})$.

C_{10} in combination with the unit clause C_9 yields $\overline{y}@0$. Under this assignment, the clause C_6 is unit, forcing us to assign $\mathsf{0}$ to z, which conflicts with clause C_7. Note that we obtained this conflict without making any decisions, i.e., we found a conflict at decision level zero. Accordingly, the formula is unsatisfiable.

Exercise 8 *Use the approach described in Section 1.3.6 to construct a resolution refutation proof for the formula presented in Exercise 7.*

Exercise 9 *Find an unsatisfiable core of the formula*

$$(y) \cdot (x + \overline{y} + z) \cdot (\overline{x} + z) \cdot (\overline{x} + \overline{y}) \cdot (\overline{z} + \overline{y}) \,.$$

(You are not allowed to provide the set of all clauses as a solution.)
Is your solution minimal?

Solution The set of clauses

$$\{(y), (x + \overline{y} + z), (\overline{x} + z), (\overline{z} + \overline{y})\}$$

forms a core of the formula in Exercise 9. This can be verified by means of resolution:

$$\begin{aligned}
&\mathrm{Res}((y), (x + \overline{y} + z), y) \equiv (x + z)\\
&\mathrm{Res}((x + z), (\overline{x} + z), x) \equiv (z)\\
&\mathrm{Res}((z), (\overline{z} + \overline{y}), z) =\equiv (\overline{y})\\
&\mathrm{Res}((\overline{y}), (y), y) = \Box
\end{aligned}$$

Moreover, the core is minimal, since removing any one of the clauses "breaks" the core. Note that $\{(y), (x + \overline{y} + z), (\overline{x} + \overline{y}), (\overline{z} + \overline{y})\}$ is an alternative minimal solution.

Exercise 10 *Simplify the following formula using the substitution approach described in Section 1.3.10:*

$$\begin{aligned}
w \cdot (\overline{q} + z) \cdot (\overline{q} + x) \cdot (\overline{z} + \overline{x} + q) \cdot (\overline{p} + \overline{z}) \cdot (\overline{p} + \overline{x}) \cdot (z + x + p) \cdot\\
(\overline{p} + u) \cdot (\overline{q} + u) \cdot (\overline{u} + p + q) \cdot (\overline{y} + v) \cdot (\overline{u} + v) \cdot (\overline{v} + y + u) \cdot\\
(\overline{x} + w) \cdot (\overline{v} + w) \cdot (\overline{w} + x + v)
\end{aligned}$$

Solution Note that we do not know which clauses are "definitional" (i.e., introduce functionally dependent variables). In practice, this information is often not available and inferring it is computationally prohibitively expensive. Therefore we will not attempt to do so. Instead, we start by dividing the clauses into sets according to the positive and negative occurrences of the literals as follows:

$$\begin{aligned}
S_x &= \{(\overline{q} + x), (z + x + p), (\overline{w} + x + v)\}\\
S_{\overline{x}} &= \{(\overline{z} + \overline{x} + q), (\overline{p} + \overline{x}), (\overline{x} + w)\}\\
S_y &= \{(\overline{v} + y + u)\}\\
S_{\overline{y}} &= \{(\overline{y} + v)\}\\
S_z &= \{(\overline{q} + z), (z + x + p)\}\\
S_{\overline{z}} &= \{(\overline{z} + \overline{x} + q), (\overline{p} + \overline{z})\}\\
S_p &= \{(z + x + p), (\overline{u} + p + q)\}\\
S_{\overline{p}} &= \{(\overline{p} + \overline{z}), (\overline{p} + \overline{x}), (\overline{p} + u)\}\\
S_q &= \{(\overline{z} + \overline{x} + q), (\overline{u} + p + q)\}\\
S_{\overline{q}} &= \{(\overline{q} + z), (\overline{q} + x), (\overline{q} + u)\}\\
S_u &= \{(\overline{p} + u), (\overline{v} + y + u), (\overline{q} + u)\}\\
S_{\overline{u}} &= \{(\overline{u} + p + q), (\overline{u} + v)\}\\
S_v &= \{(\overline{y} + v), (\overline{u} + v), (\overline{w} + x + v)\}\\
S_{\overline{v}} &= \{(\overline{v} + y + u), (\overline{v} + w)\}\\
S_w &= \{(w), (\overline{x} + w), (\overline{v} + w)\}\\
S_{\overline{w}} &= \{(\overline{w} + x + v)\}
\end{aligned}$$

Then, for each pair of sets $S_\ell, S_{\overline{\ell}}$, we derive all possible resolvents and drop the resulting tautologies. If the resulting set of clauses $\mathrm{Res}(S_\ell, S_{\overline{\ell}}, \ell)$ is smaller than $S_\ell \cup S_{\overline{\ell}}$, we replace the clauses $S_\ell \cup S_{\overline{\ell}}$ with $\mathrm{Res}(S_\ell, S_{\overline{\ell}}, \ell)$. Otherwise, we retain the clauses $S_\ell \cup S_{\overline{\ell}}$. The set of resolvents of S_x and $S_{\overline{x}}$ has five elements:

$$\mathrm{Res}(S_x, S_{\overline{x}}, x) \equiv \{(\overline{q}+\overline{p}), (\overline{q}+w), (z+p+w), (\overline{w}+\overline{z}+v+q), (\overline{w}+\overline{p}+v)\}$$

This is one clause less than $S_x \cup S_{\overline{x}}$. Accordingly, replacing the clauses $S_x \cup S_{\overline{x}}$ with the corresponding set of resolvents reduces the size of the formula. This strategy is implemented in the SAT-solver MINISAT [ES04b, EB05].

Exercise 11 *Use the core-guided algorithm presented in Section 1.4.3 to determine the solution of the partial MAX-SAT problem*

$$(\overline{x}+\overline{y}) \cdot (\overline{x}+z) \cdot (\overline{x}+\overline{z}) \cdot (\overline{y}+u) \cdot (\overline{y}+\overline{u}) \cdot (x) \cdot (y)\,,$$

where only the clauses (x) and (y) may be dropped.

Solution Assume that the first unsatisfiable core we obtain is $\{(\overline{x}+\overline{y}), (x), (y)\}$. Accordingly, we augment the clauses (x) and (y) with relaxation variables and introduce a cardinality constraint which guarantees that at most one of these clauses is dropped:

$$(\overline{x}+\overline{y}) \cdot (\overline{x}+z) \cdot (\overline{x}+\overline{z}) \cdot (\overline{y}+u) \cdot (\overline{y}+\overline{u}) \cdot (r+x) \cdot (s+y) \cdot \sum(r,s) \leq 1$$

As illustrated in Figure 1.13, we can encode the constraint $\sum(r,s) \leq 1$ as $(\overline{r}+\overline{s})$, and we obtain the instance

$$(\overline{x}+\overline{y}) \cdot (\overline{x}+z) \cdot (\overline{x}+\overline{z}) \cdot (\overline{y}+u) \cdot (\overline{y}+\overline{u}) \cdot (r+x) \cdot (s+y) \cdot (\overline{r}+\overline{s})\,,$$

which is still unsatisfiable, since

$$\begin{aligned}
&\mathrm{Res}((\overline{x}+z), (\overline{x}+\overline{z}), z) = (\overline{x})\\
&\mathrm{Res}((\overline{y}+u), (\overline{y}+\overline{u}), u) = (\overline{y})\\
&\mathrm{Res}((r+x), (\overline{r}+\overline{s}), r) = (\overline{s}+x)\\
&\mathrm{Res}((s+y), (\overline{s}+x), s) = (x+y)\\
&\mathrm{Res}((\overline{y}), (x+y), y) = (x)\\
&\mathrm{Res}((\overline{x}), (x), x) = \square
\end{aligned}$$

Accordingly, we add additional relaxation variables to the clauses $(r+x)$ and $(s+y)$ in the next iteration of the algorithm in Figure 1.14 and obtain

$$(\overline{x}+\overline{y}) \cdot (\overline{x}+z) \cdot (\overline{x}+\overline{z}) \cdot (\overline{y}+u) \cdot (\overline{y}+\overline{u}) \cdot (t+r+x) \cdot (v+s+y) \cdot \underbrace{(\overline{r}+\overline{s}) \cdot (\overline{t}+\overline{v})}_{\text{cardinality constraints}}$$

It is now possible for the satisfiability solver to relax both clauses (x) and (y) by choosing the assignment $\{t \mapsto 1, r \mapsto 0, v \mapsto 0, s \mapsto 1\}$, for instance. Accordingly, the algorithm in Figure 1.14 reports that two clauses need to be dropped to make the formula satisfiable.

Exercise 12 *Use the algorithm presented in Section 1.4.4 to derive all minimal correction sets for the unsatisfiable formula*

$$\overbrace{(x)}^{C_1} \cdot \overbrace{(\overline{x})}^{C_2} \cdot \overbrace{(\overline{x} + y)}^{C_3} \cdot \overbrace{(\overline{y})}^{C_4} \cdot \overbrace{(\overline{x} + z)}^{C_5} \cdot \overbrace{(\overline{z})}^{C_6} \; .$$

Solution (This example is borrowed from [LS08].) Due to the prioritisation of unit clauses, the first unsatisfiable core reported by the satisfiability checker is $\mathrm{UC}_1 \equiv \{(x), (\overline{x})\}$. By adding relaxation variables to all clauses of this core and by constraining the respective relaxation literals, we obtain the formula

$$(r_1 + x) \cdot (r_2 + \overline{x}) \cdot (\overline{x} + y) \cdot (\overline{y}) \cdot (\overline{x} + z) \cdot (\overline{z}) \cdot (\overline{r_1} + \overline{r_2})$$

Since dropping the clause $(\overline{x})$ does not yield a satisfiable instance, the AllSAT procedure returns C_1 as the only MCS of size one. Accordingly, we block the corresponding assignment by adding the blocking clause $(\overline{r_1})$:

$$(r_1 + x) \cdot (r_2 + \overline{x}) \cdot (\overline{x} + y) \cdot (\overline{y}) \cdot (\overline{x} + z) \cdot (\overline{z}) \cdot (\overline{r_1} + \overline{r_2}) \cdot (\overline{r_1})$$

and obtain a new core $\{(\overline{r_1}), (r_1{+}x), (\overline{x}{+}y), (\overline{y})\}$. Accordingly, $\mathrm{UC}_2 = \{C_1, C_2\} \cup \{C_1, C_3, C_4\}$, and we obtain the instrumented formula

$$(r_1 + x) \cdot (r_2 + \overline{x}) \cdot (r_3 + \overline{x} + y) \cdot (r_4 + \overline{y}) \cdot (\overline{x} + z) \cdot (\overline{z}) \cdot (\overline{r}_1) \cdot \sum(r_1, r_2, r_3, r_4) \leq 2$$

The AllSAT algorithm determines all minimal correction sets for this formula. Note that the clause $(\overline{r}_1)$ prevents that the algorithm rediscovers the MCS $\{C_1\}$ in this step. Since $\mathrm{Res}((\overline{r}_{)}, (r_1 + x)) \equiv (x)$, blocking C_1 yields the formula

$$(x) \cdot (r_2 + \overline{x}) \cdot (r_3 + \overline{x} + y) \cdot (r_4 + \overline{y}) \cdot (\overline{x} + z) \cdot (\overline{z}) \cdot \sum(r_1, r_2, r_3, r_4) \leq 2\,,$$

which is unsatisfiable. We obtain the new core $\{C_1, C_5, C_6\}$ and execute the third iteration of the algorithm with $\mathrm{UC}_3 = \{C_1, C_2, C_3, C_4\} \cup \{C_1, C_5, C_6\}$. The corresponding instrumented and constrained version of the original formula is

$$(r_1 + x) \cdot (r_2 + \overline{x}) \cdot (r_3 + \overline{x} + y) \cdot (r_4 + \overline{y}) \cdot (r_5 + \overline{x} + z) \cdot (r_6 + \overline{z}) \cdot \\ \sum(r_1, r_2, r_3, r_4, r_5, r_6) \leq 3$$

In this iteration, we obtain the MCSes $\{C_2, C_3, C_5\}$, $\{C_2, C_3, C_6\}$, $\{C_2, C_4, C_5\}$, and $\{C_2, C_3, C_6\}$. Adding the corresponding blocking clauses to Instrument(F) results in an unsatisfiable instance and the algorithm terminates.

Exercise 13 *Derive all minimal unsatisfiable cores for the formula presented in Exercise 12.*

Solution The set of MCSes for the formula in Exercise 12 is

$$\{\{C_1\}, \{C_2, C_3, C_5\}, \{C_2, C_3, C_6\}, \{C_2, C_4, C_5\}, \{C_2, C_3, C_6\}\}\ .$$

We construct the corresponding minimal hitting sets as follows:

MCSes(F)	C_1	C_2	C_3	C_4	C_5	C_6
$\{C_1\}$	×					
$\{C_2, C_3, C_5\}$		×	×		×	
$\{C_2, C_3, C_6\}$		×	×			×
$\{C_2, C_4, C_5\}$		×		×	×	
$\{C_2, C_3, C_6\}$		×		×		×

Hitting sets: $\{C_1, C_2\}, \{C_1, C_3, C_4\}, \{C_1, C_5, C_6\}$