Proof Support for Hybrid Systems Verification, IV

Hybrid Systems and other Applications

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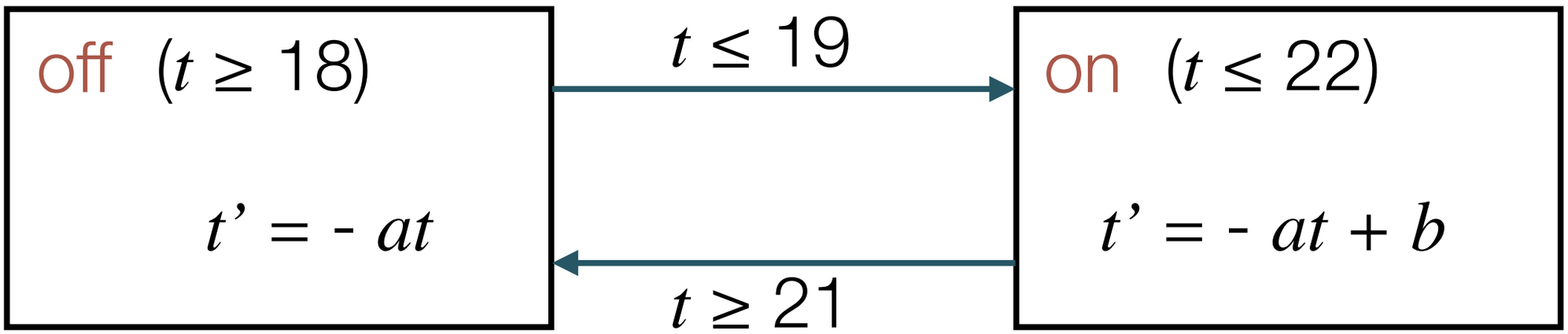
Hybrid systems

- A *hybrid system* has a compound state containing both **discrete** and **continuous** parts. The discrete part is a finite-state machine where each state has continuous dynamics.

- Simple examples:
  - room heater with thermostat
  - a water tank with inlet and outlet pipes with valves
  - an autopilot with multiple modes
  - any sort of physical process controlled by a computer

Room heater with thermostat



target temperature: 20°C

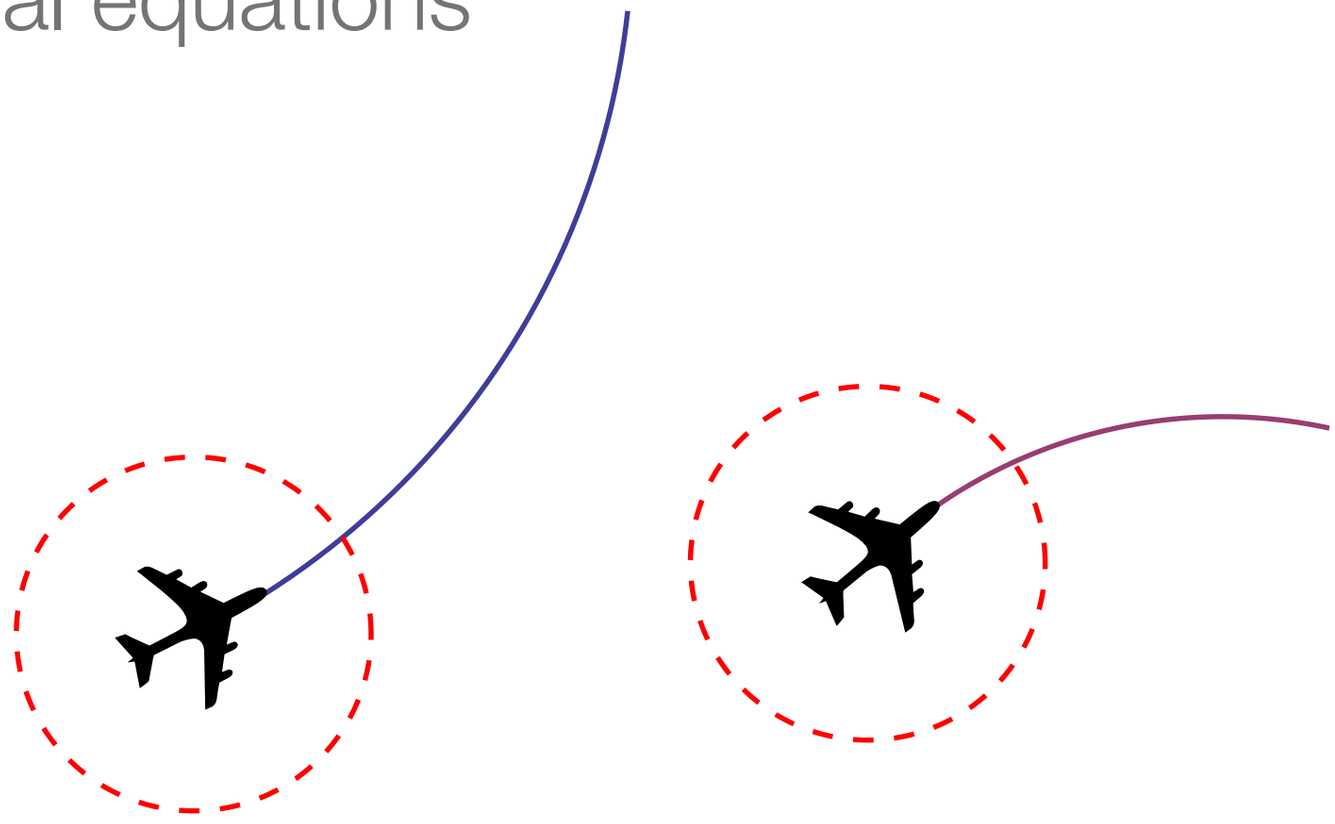
switches at 19 and 21 to prevent oscillation

dynamics given by differential equations

Example from John Lygeros, Lecture Notes on Hybrid Systems, 2004.

1: A collision avoidance problem

- two aircraft, *x* and *y*, flying in two dimensions (sorry)
- first studied by Platzer (2010), using KeYmaera
- MetiTarski treatment due to Denman, using *closed-form solutions* of the differential equations



The system of differential equations for aircraft *x*

x_1 denotes *position* in the **first** coordinate; d_1 denotes *velocity*

$$x_1'(t) = d_1(t) \qquad x_2'(t) = d_2(t) \qquad d_1'(t) = -\omega d_2(t) \qquad d_2'(t) = \omega d_1(t)$$
$$x_1(0) = x_{1,0} \qquad x_2(0) = x_{2,0} \qquad d_1(0) = d_{1,0} \qquad d_2(0) = d_{2,0}$$

x_2 denotes *position* in the **second** coordinate; d_2 denotes *velocity*

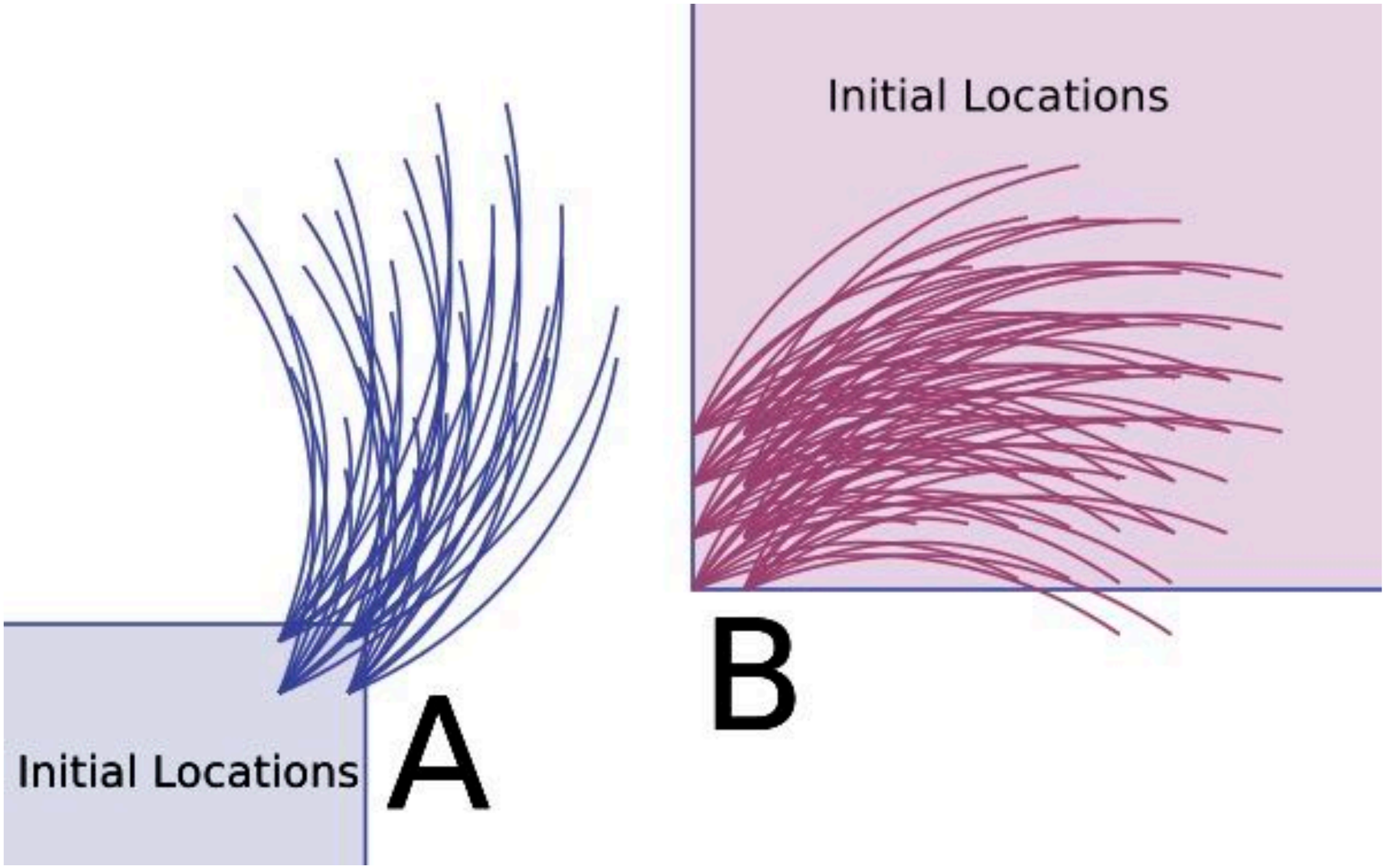
… and the closed-form solution

$$x_1(t) = x_{1,0} + \frac{d_{2,0}\cos(\omega t) + d_{1,0}\sin(\omega t) - d_{2,0}}{\omega}$$

$$x_2(t) = x_{2,0} - \frac{d_{1,0}\cos(\omega t) - d_{2,0}\sin(\omega t) - d_{1,0}}{\omega}$$

Solutions of linear differential equations frequently involve simple combinations of transcendental functions.

Possible paths of the two aircraft



The desired safety property

Two aircraft following those equations…

subject to certain other parameters…

must maintain a *safe distance*, *p*:

$$(x_1(t) - y_1(t))^2 + (x_2(t) - y_2(t))^2 > p^2$$

The resulting MetiTarski problem

```
fof(airplane_easy,conjecture,
  (! [T,X10,X20,Y10,Y20,D10,D20,E10,E20] :
    (
      ( 0 < T & T < 10 & X10 < -9 & X20 < -1 & Y10 > 10 & Y20 > 10 &
        0.1 < D10 & D10 < 0.15 & 0.1 < D20 & D20 < 0.15 &
        0.1 < E10 & E10 < 0.15 & 0.1 < E20 & E20 < 0.15 )
      =>
      ( (X10 - Y10 - 100*D20 - 100*E20 + (100*D20 + 100*E20)*cos(0.01*T)
        + (100*D10 - 100*E10)*sin(0.01*T))^2 +
        (X20 - Y20 + 100*D10 + 100*E10 + (-100*D10 - 100*E10)*cos(0.01*T)
        + (100*D20 - 100*E20)*sin(0.01*T))^2 )
      > 2 )
  )
).
include('Axioms/general.ax').
include('Axioms/sin.ax').
include('Axioms/cos.ax').
```

Remarks about this proof

- originally required 924 seconds (using Z3)
- can take as little as 30 seconds, depending on setup
- 9 variables! (the most ever in a MetiTarski proof)

William Denman attended Marktoberdorf in 2012!

2: Model checking by *qualitative abstraction*

- **Over-approximate** the (continuous) transition relation by a *discrete* state transition system
- Denman's QUANTUM abstracter uses MetiTarski
  - to **remove** infeasible abstract states
  - to **validate** abstract transitions
- It's based on HybridSal (Tiwari, SRI); the abstract models are checked using SAL (Symbolic Analysis Laboratory)

Model checking alongside theorem proving!

Building the model requires hundreds of MetiTarski runs

The actual certification comes from model checking

Failed proofs make the model more complicated,
but aren't fatal

Contrast with other approaches where a conjunction of properties MUST be proved

Self-driving car performing a curved turn

one of the 1190 “feasibility” problems

```
fof(stdin, conjecture,
    (![X,TT,G] : (~(G<1.5 & G>-1.5 & TT>=0 & TT<6 & X + 2>0 & X - 2>0 & X
- 0.2*cos(TT - 1.5) + 0.25>0 & X - 0.2*sin(TT) - 0.5>0 & G=0 & G -
0.5>0)))).
```

one of the 2123 “transitions” problems

```
fof(checkTransition, conjecture,
    (![X,TT,G] :(G<1.5 & G>-1.5 & TT>=0 & TT<6 & X + 2>0 & X - 2>0 & X -
0.2*cos(TT - 1.5) + 0.25>0 & X - 0.2*sin(TT) - 0.5=0 & G>0 & G - 0.5=0
 => (-2*sin(G) + 0.2*sin(TT - 1.5) > 0 |
     -2*sin(G) + 0.2*sin(TT - 1.5) = 0)))).
```

QUANTUM verifies *unbounded* safety properties,
unlike many other hybrid system verifiers

Other examples include a ball
bouncing on a sine curve and a
system with two water tanks

3: Error analysis for numerical computation

```
fof(atan_error_analysis_5,conjecture,
    ! [X] : ((abs(X) <= 1/30) =>
        abs (arctan(X) -
             (X - 11184811/33554432 * X^3 - 13421773/67108864 * X^5))
        <= (1/2)^8) ).
```

Typically a narrow range is checked

Bespoke applications could need any precision

could have 100s of cases, verifying *table entries*

Such problems are trivial for MetiTarski!

But can MetiTarski be trusted? Some issues

- *Arithmetic simplification*: reducing polynomials to canonical form, using ad hoc code

- *Specialised axioms* giving upper or lower bounds of special functions

- Reliance on an external RCF decision procedure

but we do get machine-readable proofs!
Could they be validated independently?

Algebraic Simplification

Translation to canonical form

Obvious cancellation laws

$$\left(\frac{x}{y}\right)\frac{1}{(x+\frac{1}{x})} = \frac{x^2}{y(x^2+1)}$$

Transformation of quotients!

Independent reconstruction in an should be straightforward...

Verifying the axioms

- The bounds for $\sqrt{}$, $\mathbf{sin}$ and $\mathbf{cos}$ already verified in Isabelle (infinite families of bounds).

- No general Isabelle theory of continued fractions yet, but the bounds used in MetiTarski have been verified:

- For $\mathbf{exp}$ and $\mathbf{ln}$, all including the most complicated

- For inverse tangent, verified all but two bounds

The weakest link: the decision procedure

- The best-known procedure (*cylindrical algebraic composition*) is complicated and requires efficient computer algebra tools.

- Real quantifier elimination is *doubly exponential* in the number of variables (Davenport and Heintz, 1988)

- Few implementations of any sort exist; fewer justify their answers with any sort of **evidence**. (Mainly in HOL-Light.)

  - *Hörmander's procedure:* far too slow

  - *Sum-of-squares methods*

Must we trust the decision procedure?

- During search, MetiTarski may call the decision procedure hundreds of times, also to *discard redundant clauses*.

- We only need to trust calls appearing in the proof, but there could still be dozens!

- These are specific polynomial inequalities, which could be checked by other means (not necessarily deductive).

- There are multiple, independent implementations of CAD.

Summary

Hybrid systems can be verified on the basis of closed-form solutions of their dynamics

MetiTarski can also be used to support model checking by qualitative abstraction

The dynamic behaviour can be specified using transcendental functions

Simple numerical algorithms can also be verified!

The trust issues are significant but tractable.

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